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RESEARCH ARTICLE

# A Novice Teacher Researcher's Action Research Project: Posing Problems to Promote Concepts of Graphs in Calculus

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A novice mathematics teacher researcher (TR) conducted an "interactive action research" (AR) to determine what problems would be most beneficial to teach her students about creating graphs based on a function's attributes. After a number of trials that included adjusting her goals, the TR successfully designed problems appropriate for her goals. This paper describes the problem-posing process the TR used to derive the problems, and which include the four steps described in the literature: i) plan the problem, ii) pose it, iii) solve it, and iv) organize and complete it (see Güveli, 2015) plus an additional overall step added by this author, v) develop awareness of common perceptions (and misconceptions) that students have with respect to graphing. The contribution of this study is twofold. The first is the theoretical model of a five-step AR process, which can be used to guide TRs when conducting a mathematics posing problem AR: mathematical objective, source of inspiration, concerns related to formulation, mathematical uncertainties, and decisions taken. The second is that it demonstrates how TR's formative assessment of the student's solutions can improve her problem-posing heuristics and guide her to adjust her didactic goal(s). In addition, this paper documents her professional development on two aspects: developments and transitions in her thinking, and her development in skills required for reaching a didactic or mathematical goal.

**Keywords:** action research, problem posing, novice teachers, calculus, graphical representation, graphing competence

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## Theoretical background

### *From teacher to researcher*

Loughran (2014) highlighted the value that examining one's own teaching practices has for teachers or pre-service teachers and how investigating one's own teaching practices can illuminate both effective and ineffective teaching methods. Such "action research" (AR) offers a valuable professional learning experience (Souto-Manning, 2012). In this respect, AR can contribute to the generation of both local and public knowledge about teacher education.

Requiring pre- or in-service teachers to fill the roles of teacher researchers (TR) and engage in AR during their practical training moves their education away from the educator-centered model (where the educator "communicates" information "to" the student) towards a more student-centered approach, thus transforming pre-service teachers from receivers of pedagogical knowledge from "higher" authorities into creators of such knowledge (Cochran-Smith & Lytle, 2009). Knowing how to conduct proper research is an important tool that can extend beyond the teacher-education classroom, allowing in-service teachers to continue their professional development even after completing their formal coursework and providing them opportunities to improve teaching and learning (Lysaker & Thompson, 2013). Snow-Gerono (2005) found that engaging in AR led teachers to change not only their classroom practices but also their attitudes towards teaching, and that even small-scale research projects will improve their understanding of the research process (Gray, 2013), their selected disciplines(s) of study (Goodnough, 2010), and even about themselves.

Other studies have shown that AR projects lead teachers to improve their teaching skills and to better understand their beliefs and attitudes toward themselves as teachers and toward their students as learners. For example, Davis, Clayton and Broome (2018) investigated novice teachers' responses to an AR project conducted during their practical semester and found that the teachers' pre-existing identities as researchers influenced their research process and that a positive research experience had a positive impact on their teaching practice.

### *Problem posing by teachers*

Posing problems for a mathematical activity involves many cognitive processes (Cai et al., 2020). Indeed, recent research has highlighted the importance of posing problems in terms of the professional development of mathematics teachers (Chen & Cai, 2020; English, 2020; Lee, Capraro & Capraro, 2018). For example, English (2020) found that problem posing (PP) helps teachers become more proficient mathematically.

A study by Lee, Capraro and Capraro (2018) found that teachers tended to avoid lessons in which they had to pose problems; therefore, teaching them how to go about posing problems should be a part of the teacher-training curriculum to ensure the proper professional development of quality mathematics instruction. Teachers require applicable pedagogical content knowledge and strategy to provide their students with efficient learning opportunities, and being able to pose problems is a very important skill (Lee et al., 2018; 2019). Nevertheless, little has actually been written about how to teach mathematics teachers to pose problems to teach mathematics more effectively (Cai et al., 2020). Other scholars have also emphasized that the topic of PP should be included in the teacher training process (e.g., Crespo & Harper, 2019; Leavy & Hourigan, 2019).


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There are a multitude of other studies that have examined problem-posing skills in teachers. A study by Chapman (2012) examined the skills of teachers to produce original verbal problems or rearrange existing ones and concluded that teachers had difficulties as a result of their limited experience in PP. Studies carried out by Ellerton (2013) and Rosli et al. (2015) examined the experiences of teachers in training programs that integrated problem solving and PP. The former determined that teachers found PP more difficult than problem solving while the latter observed that teachers tended to pose problems that were similar in structure to problems, they were already familiar with and that they failed to find original ways to present problems so as to present the concept under study from a different approach. Chapman (2012) and Ellerton (2013) both concluded that, for the most part, teachers had difficulty posing complex problems that require extended content and high-level thinking skills. Many other studies have shown that the majority of teachers find that posing problems is a challenging task (e.g., AydogduIskenderoglu, 2018; Bayazit & Kırnap-Dönmez, 2017; Kar, 2016; Ulusoy & Kepceoğlu, 2018; Chapman, 2012; Leavy & Hourigan, 2019; Xie & Masingila, 2017).

The present study aims to examine a teacher's problem-posing process and skills to produce a model that can effectively ease teachers' difficulties when posing problems. The model focuses on how to properly determine the didactic goals for a specific class: that is how, through analyzing the students' solutions, the teacher can decide whether the didactic goals need altering and as a result, adjust the problems posed to make them more relevant both to her didactic goal and her students' needs.

### *Drawing graphs of functions in calculus*

The topic of concern in this particular study was in the field of calculus. The interpretation of functions and plotting their graphs are essential skills for all mathematics students alongside the ability to understand what different representations express. In fact, it is one of the mathematical competences mentioned in the national educational standards for mathematics (NCTM, 2000) Dreyfus & Eisenberg, 1986; Eisenberg & Dreyfus, 1991). However, dealing with graphs of functions can be difficult and easily lead to misconceptions, many of which are noted in the literature (e.g., Nitsch, 2015; Leinhardt et al., 1990; Clement, 1985; Bell & Janvier, 1981) and include the graph-as-picture misconception, slope-height confusion, and interval-point confusion. Moreover, what students think a function is or how the graph of a function should appear (concept image) does not always correspond to the definition of a function that the students have (concept definition) (Tall & Vinner, 1981). Oftentimes, these errors/misconceptions are not clearly discernable by the teacher, yet they need to be overcome as soon as possible: once erroneous notions become embedded in the student's mind, they become very hard to overcome and correct (Nitsch, 2015) and may follow the learner even to higher education.

Calculus requires a particularly in-depth understanding of subject-related concepts and the broad relationships between the concepts' representations. Fostering such understanding necessitates solving increasingly challenging problems that demonstrate explicit connections between concepts (and between different representations of the concept) and that offer unique insights for specific examples and families of functions (Park, 2015).

Students who are asked routine questions in calculus usually offer predictable, routine solutions. Offering non-routine problems can underscore difficulties students may have in understanding particular concepts and the transitions between them (De Bock, van Dooren, & Verschaffel, 2015; Chang, Cromley, & Tran, 2016). For example, Park's (2015) study of how teachers teach the concept of derivatives found that their discussions about the derivative as a function made limited use of graphical solutions.

### *Perceptions and misconceptions about graphs*

David, Roh, and Sellers (2019) found that students think about graphs two ways: "value-thinking" and "location-thinking." One of the key characteristics of value-thinking is distinguishing between the output of a function and the point on a graph, and students who engage in value-thinking label points as ordered pairs, e.g., (a, f(a)), and speak about points as representing both input and output values simultaneously. Location-thinking implies thinking about graphs by relying on the spatial locations of points on the Cartesian plane. Students engaged in location-thinking focus on the location of points, but the values of the coordinates may or may not be a part of their reasoning.

Tall (2010) developed an intervention designed to highlight the relationship between the visual representation of the continuity of a function with its analytic definition. Moore and Thompson (2015) defined two new definitions: static-shape thinking and emergent-shape thinking. Static-shape thinking involves treating a graph as an object in and of itself, essentially considering the graph as "a piece of wire" (graph-as-wire) and entails assimilations and actions based on perceptual cues and the perceptual shape of a graph. Emergent-shape thinking involves simultaneously understanding a graph as what is made (a trace) and how it is made (covariation) and entails assimilating a graph as a trace in progress, as opposed to a static object.

Students may have many misconceptions about how to sketch the graphs of functions (Koray & Bal, 2002). Clement (1985) mentioned two types of common misconceptions about graphs of functions: "treating the graph as a picture" and "slope-height confusion." Glazer (2011) emphasized that it is important to demonstrate graphs of functions to students and discuss both how the graphs are created and − perhaps even more crucial − how to interpret a graph. This may suggest that students' misconceptions may be amplified by the inappropriate use of graphs in high school and university textbooks (Kajander & Lovric, 2009), the extent of the teachers' pedagogical content knowledge when teaching (Rubel, 2002), the teachers' use of inappropriate visual materials (Mudaly & Rampersad, 2010), and inconsistencies that are present in the student's mind when dealing with interrelated mathematical concepts (Tall, 1990). As a result, traditional instructional methods may foster misconceptions (Marek, Cowan, & Cavallo, 1994; Ubuz, 1999). Clearly, the combination of all these may negatively influence students' understanding and eventually become a serious obstacle to learning.

Neidorf and colleagues (2020) found that a higher percentage of eighth-grade students were able to translate a graphical representation into a verbal description but not into an algebraic equation. This might mean that students are able to understand the relationship represented by a graph of a line but are not well-versed in the symbolic representation of that same line, what each symbol means, and how they are related. Instruction needs to focus on these aspects, with an emphasis on an understanding that goes beyond using equations to find the value of one variable when another is given.

While graphing competence includes both interpretation and construction, most of the studies have focused on interpretation and little is known about students' conceptions and alternative conceptions regarding graph construction (Glazer, 2011).

In the present study, the teacher was aware of the various pedagogical theories described above, in particular the ideas put forth by Neidorf and colleagues and Glazer. She thus tried to mitigate any difficulties by posing problems that had specific wording that would allow identifying the presence of misconceptions in the students' thinking (as shown by how they derived their solutions) and then offering a number of similar problems to consolidate understanding.

## Research goal

To examine and analyze a teacher's problem-posing process and thus better understand what practices a teacher must undertake in order to successfully pose problems to reach a specific mathematical didactic goal. In other words, how can a TR carrying out a problem-posing AR refine her goals and methods to improve her students' understanding of how graphs represent functions in calculus?

## Method

### *Background*

The participant in this study was a novice teacher in the final stage of an M.Ed. program and experiencing her first year of teaching: grade-twelve mathematics (18 students in the class), five 45-minute lessons per week, for one semester. She had not as yet had any experience in posing problems nor had she learned anything about posing problems to reach a specific goal.

As part of her M.Ed. program, she was to design and carry out an AR of her choosing, during which she was to assess how to better reach her specific didactic goal, which was for her students to develop a firm basis in creating and understanding the concept of visual representations (i.e. graphs) of calculus functions based on the properties of the function and be able to solve calculus problems through (analyzing a function's) graphic properties.

The TR's research goal was to analyze her students' answers to the problems she posed and decide what she might have to do to further her didactic goal. For the week's first lesson, she posed a problem and designed an assessment lesson around it. She noted and, immediately after each lesson, analyzed the students' solutions to determine their understanding of the concepts. She documented her thinking before and after each lesson in her reflection journal and then used that information to design and pose "improved problems" for the following lessons. This continued for the entire semester.

The TR wrote down her plans in a journal before the lessons and her reflections thereafter, and also noted how she processed the analysis of the implementation of the problems posed, her observation of the solutions her students obtained, and her conclusions thereof. This author served as the mentor for the TR and documented the entire process in a "mentor's journal."

### *Data collection*

The two journals were the primary source of data. The secondary source of data was the documentation of the mentoring sessions between this author and the TR two meeting per lesson: one to guide the TR about how to think about posing the problem and the second after posing the problem. The purpose of the data analysis was to focus on the two processes: the cyclical process of an AR (planning, implementation, observation, and reflection, see Gilbert & Newberry, 2004) and the problem-posing process (planning, posing, solving, organizing, see Güveli, 2015, plus awareness of the students' perceptions). A third data resource was the documentation of the math tasks that the TR made during the lessons. However, in the present study, we will not focus on the analysis of the solutions that the teacher did, but on her general understanding of her students' perceptions that led her to design the next problem.

### *Data analysis*

Each journal was examined on three levels that were compared (triangulation): i) documentation, ii) reflection, and iii) analysis and interpretation.

The transcripts (from both journals) were divided into "episodes" and reread twice more, each time seeking insight from a different aspect. The first was the teacher's general method and heuristic for posing problems, which could be divided into five sub-categories: mathematical objective (the goal the teacher set for herself in posing the problem), source of inspiration (the source of knowledge that inspired the subject and the style of the problem posed), concerns related to formulation (thoughts about wording.), mathematical uncertainties (deliberations and doubts the teacher expressed about the problem that were expressed during the process), and decisions taken (after deliberation).

The second aspect was to derive insight about the development of the TR's problem-posing skills that would allow her to effectively reach her didactic goal. A list of repeated actions was generated (open coding, see Corbin & Strauss, 2014) for each aspect and was broken down into six subcategories: i) modifying an existing problem to achieve a similar goal, ii) modifying an existing problem to achieve different goal, iii) formulating and posing an original problem based on an existing graph\goal\theory, iv) formulating and posing an original problem to understand her students' perceptions of a feature or concept of graphs, v) formulating and posing a problem to change her students' perceptions about a feature or concept of graphs, and vi) being able (i-v together) to accurately formulate a problem to achieve a specific mathematical goal. The data was then assembled into a chart. Random episodes could then be analyzed and examined as to how they fit into the scheme of categorization.

## Results

The findings are divided according to the two aspects of the TR's development of problem-posing mentioned above (AR process and developing skill in PP).

*Problem posing as action research*: Below are the four problems that the TR posed over the first four weeks with an analysis of the five-step AR process found and defined above and followed by an analysis of the skills the TR developed as a result and that led to the subsequent problem.

### Week 1

Mathematical objective: Drawing a graph based on a list of attributes.

Source of inspiration: A similar problem solved by the TR in a different course using a technological tool (from the step center http://visustep.com/).

Concerns related to formulation: Maintaining similarity to the wording observed in the original problem.

Mathematical uncertainties: Will the students understand the meaning of the conditions of the problem?

Decisions taken: Change the wording of the conditions for formulation and concepts to those that students are familiar with from previous problems.

**Task No. 1: Conditions for a function**

Following are some conditions that may or may not apply to a function:

i. It has two different horizontal asymptotes.
ii. It has a vertical asymptote.
iii. Its derivative does not vanish (no discontinuity).
iv. It is positive for all *x*.

1) Is there a function that meets any two conditions? Yes / No.
   If so, indicate which two, find a suitable function, and draw its graph. If not, explain why.
2) Is there a function that meets any three conditions? Yes / No.
   If so, indicate which three, find a suitable function, and draw its graph. If not, explain why.
3) Is there a function that meets the first condition plus two others? Yes / No.
   If so, indicate the two additional conditions, find a suitable function, and draw its graph. If not, explain.
4) Is there a function that meets all four conditions? Yes / No.
   If so, find a suitable function and draw its graph. If not, explain.

The starting point and inspiration for the problem was one with the mathematical goal "to be able to draw a function that includes at least two conditions." The TR understood that to meet her teaching goals, she needed to pose a problem whose solutions could be presented graphically.

Next, she looked at the problem through her students' eyes to better understand what her students might offer as solutions. This led her to revise the wording to better suit the heterogeneity of her class.

At this point, it was clear that she had developed/used three skills: i) deciding and focusing on assessing how a graph of a formula that one was unfamiliar with may be drawn based on a list of attributes; ii) designing a structure to change the original dynamic problem to a new, static one (this is termed semi-structural PP [Stoyanova and Ellerton, 1996]); and iii) modifying wording to make it appropriate for her student's knowledge (she did this twice). As Kılıç (2013b) has stated, the problem-posing framework can be defined as semi-structured when the teacher deals with revising and transforming a problem, and structured when the teacher deals with comprehending and selecting a problem. Thus, this first problem she posed was derived in both semi-structured and structured stages.

### Week 2

Mathematical objective: Estimating the root and exponent attributes of a graph without resorting to algebraic operations.

Source of inspiration: A problem presented in the national exams book.

Concerns related to formulation: Nothing specific.

Mathematical uncertainties: At first she thought of asking for the absolute value of the exponents, but she feared her students would not understand its graphical significance.

Decisions taken: Pose an introductory question to assist solving the problem. Remind students of the problem they solved in the "exam book" and pose similar ideas

**Task No. 2: Exponential functions**

1) What is the difference between x and $\sqrt{x^2}$ ?
2) Based on question 1, and based on question 3 from Exam no. 13, sketch the following functions: $f_1 = 3^{\sqrt{x^2}}$ , $g_1 = 9^{\sqrt{(x-1)^2}}$.
3) When is $f_1 > g_1$ ?

Compare $f$ and $f_1$. Compare $g$ and $g_1$. Compare $f_1$ and $g_1$. (Which are similar? Which are different? Which are equal?)

The TR was careful to ensure that the problem was worded similarly to that in the exam book, so as to be familiar to the students. The uniqueness of the problem was that it was a complex function that cannot be easily solved using algebraic procedures. As a result, students are forced to draw the graph based on attributes and not by algebraic actions.

When posing this problem, she developed two skills: i) posing sub-questions, which demonstrated her awareness to her students' knowledge and her students' tendencies when problem solving; and ii) designing the wording to force her student to sketch graphs as an answer. Note that for **Task 1**, students could avoid sketching a graph and write an algebraic solution.

Similar to the first task, here (**Task 2**) too she was inspired by a problem she had solved previously, so this strengthened her skill of using a "structured" problem (Kılıç, 2013b). This structure can be explained by teachers' knowledge of patterns and problem-solving strategies, her imagination and creativity, experience with problem solving (Chapman, 2012), and her general educational experience (Tichá & Hošpesová, 2012).

### *Week 3*

Mathematical objective: Mathematical content learned in class as an exponential function.

Source of inspiration: Questions in mathematics textbooks with similar graphical solutions to what she desired. She chose a two-point graph so the derivative would have two points of intersection with an x-axis.

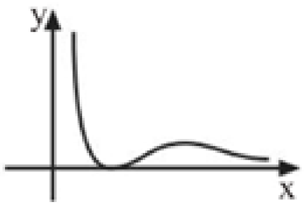
Concerns related to formulation: A clear formulation that includes concepts that students are familiar with from the lessons with a focus on sketching the graph.

Mathematical uncertainties: She did not emphasize in class the connection between the definition of domain in the function and the definition of domain in the derivative.

Decisions taken: Ask a preliminary question: What is the definition of the domain of a function's derivative? Then ask to draw the graph so as to define the graphical perceptions.

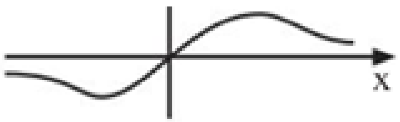
***Task No. 3: Defining the derivative function***

Below is a graph representing function $f(x) = \frac{(lnx)^2}{x^2}$. Find the domain of the derivative graph, f′(x).




Sketch the graph of f′(x).

Below is a graph representing function $(x) = \frac{2x}{(x^2+4x)}$. Sketch the function $\int g(x)$.




The TR's considerations in posing the third problem were mathematical. She found a graph with properties that correspond to the content she wanted to teach and posed algebraic expressions and questions accordingly. This is the first problem that she wrote that was not based on another problem she had encountered earlier but rather guided by the mathematical purpose to obtain solutions as specific graphs.

By this time, she had studied the literature and had learned that PP can link algorithmic thinking and conceptual knowledge (Abramovich, 2015). Therefore, she tried to express the problem in such a way as to offer the maximum number of opportunities to bridge her students' perceptions of the algebraic expression to an appropriate graph and vice versa. This was a problem for which students had to do only a minimum number of algebraic operations to gain a deep conceptual understanding of the graphs involved.

The teacher was now also familiar with the theory that students think about graphs using value-thinking and location-thinking (David, Roh, & Sellers, 2019). She therefore made a point to pay attention to the students thinking process regarding the graph and formulated a question that would present the students with the properties of the derived graph.

In week three, the skills she developed were i) the ability (and courage) to compose an entirely new problem for an existing graph for a very specific purpose and ii) a new understanding of how students can misunderstand the properties of a derivative graph.

At this point, the teacher compiled a list of mistakes and misconceptions that she expected to find in the students' solutions.

### *Week 4*

Mathematical objective: To explore and discover knowledge about derivatives of trigonometric functions.

Source of inspiration: An idea she heard from another teacher in the school.

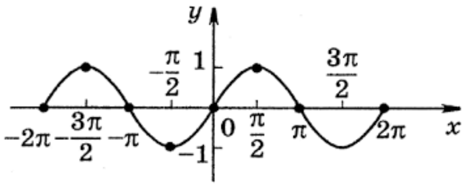
Concerns related to formulation: An attempt to formulate a mathematical language that includes important concepts.

Mathematical uncertainties: None. She was confident that the idea was appropriate for developing a concept of a derivative graph structure.

Decisions taken: To write the problem as soon as she had decided upon it. She did not need to adapt it further.

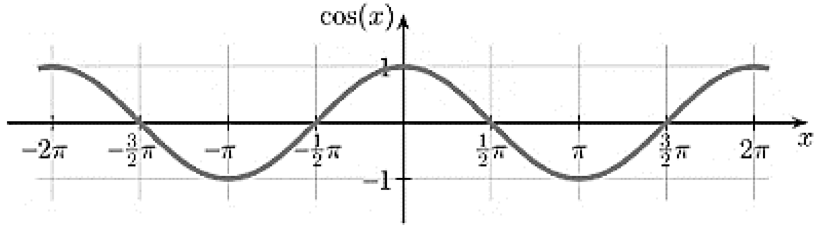
***Task No. 4: Derivatives of trigonometric functions.***

1) Below is a graph of function $y = sinx$ with domain -2π≤x≥2π.



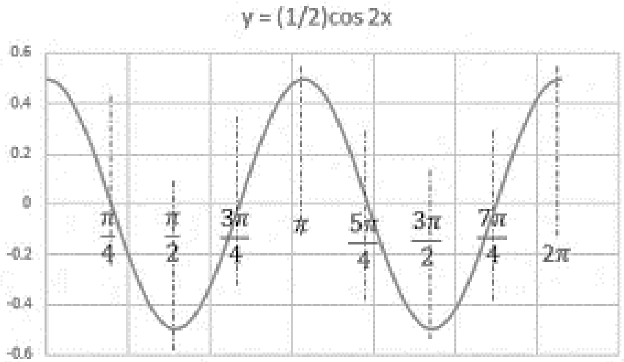

From your knowledge about the relationship between function and derivative, draw the derivative of $y = sinx$ for the given domain. What is the derivative's algebraic expression?

2) Below is a graph of function $y = cosx$ in the domain of -2π≤x≤2π.




From your knowledge about the relationship between function and derivative, draw the derivative of $y = sinx$ for the given domain. What is the derivative's algebraic expression?

3) Below is a graph of function $y = \frac{1}{2cos2x}$ with a certain domain.




From your knowledge about the relationship between function and derivative, sketch a graph of the integral of the equation in the given domain. Based on this, what is the *antiderivative* of $y = \frac{1}{2cos2x}$?

The TR was confident about the problem she posed from beginning to end and considered it successful because the students presented solutions according to her expectations.

In her opinion, this problem revised the mathematical concepts of **Task 3** using a different function. Thus, she was sure of its quality and that it was suited to her purposes.

The skills used in this case were i) learning from a colleague's idea (i.e. not only relying on her own experience); ii) designing a task following the analysis and understanding of students' errors in a previous task; and iii) eliminating erroneous conceptions. She understood that more than one drawing task should be required for the problem and that the graph drawn in each part of the problem builds a perception that serves to further understand the solution in the next part of the problem. According to English (2020), PP helps teachers become more proficient mathematically. This can be seen here, as the teacher clearly designed this problem based on mathematical considerations, indicating a broad understanding of the relationship between trigonometric functions and their derivatives.

In both **Tasks 3** and **4**, the teacher designed non-routine problems that were not similar to any problem in the textbooks. **Tasks 3 & 4** were deliberately formulated to be non-routine so that the students would not be able to offer routine solutions (see above) and to ensure that they understood the reasons behind whatever actions they chose. In addition, the solution necessarily requires an understanding of the meaning of the graph of the original function and its relation to the graph of the derived function.

What precisely was non-routine about these tasks? In **Task 3**, the student was required to understand the relationship between an exponential function graph and its derivative and the relationship between a rational function graph and its derivative graph in order to draw the representational graph. In **Task 4**, the students had to learn and understand the derivation of trigonometric functions they had not encountered before. In other words, they had to understand the properties of the given trigonometric functions to draw the derivative graphs.

Note that a study by Park (2015) revealed that teachers often make limited use of graphical solutions when discussing the concept of derivatives as a function in class. In this study, and as a result of her AR, the TR understood that part of the success in understanding graphs based on attributes comes from solving problems that require finding the derivative graph by analyzing the attributes of the original function. This understanding led her to pose non-routine problems that are not commonly used by mathematics teachers.

## Discussion

### *The development of the problem-posing process during AR*

**Table 1** summarizes the five steps of AR process observed during this study for each task. These steps can be linked to the five-step problem-posing process based on Güveli (although not on a one-to-one basis) as follows: "mathematical objective" and "source of inspiration" can be considered to be the "plan the problem" stage; "concerns related to formulation" parallels the "pose it" phase; "mathematical uncertainties" arose in the stage equal to the "solve it" stage; and finally "decisions taken" can be equated to the "organize and complete it" and "develop awareness of common perceptions" refer to "decisions taken." Thus, as a result of this study, I was able to equate the steps for conducting an AR with those that focus specifically on mathematical PP for specific goals. The development of the TR according to the five subcategories is summarized in **Table 1**.

**Table 1.** Developments and transitions in the teacher-researcher's thinking

| Step in problem-posing process | Development during the steps of the AR process |
|---|---|
| **Mathematical objective** | **Plan it:** From a general purpose to a precise and specific purpose. |
| **Source of inspiration** | **Plan it:** From learning from a specific problem to learning from a general idea. |
| **Concerns related to formulation** | **Pose it:** From copying wording from a given problem to independently composing wording appropriate to the didactic goal. |
| **Mathematical uncertainties** | **Solve it:** From general thoughts on student perceptions to focused thoughts on student perceptions. |
| **Decisions taken** | **Organize and complete it:** From awareness of student difficulties leading to changing the problem to the state where awareness of student perceptions are taken into consideration from the beginning of designing the problem. |

In each subcategory, the TR experienced a change in her thinking that resulted from her considerations of the suitability of the problem posed. As Kontorovich and Koichu (2009) argued, PP is rooted in two categories known from the problem-solving literature: beliefs and self-regulation. They quote Goldin (2002), who interprets beliefs as one's "multiply-encoded cognitive/affective configurations," to which the holder attributes some kind of truth value. They also define self-regulation in of Schoenfeld's (1992) terms, that is, as an account of cognitive processes aimed at assessment of an entire solution. Kontorovich and Koichu argued, these aspects are interrelated and, when considered together, help in understanding the cognitive/affective mechanism that governs a problem poser's convergence to a particular problem formulation. In this study, the "concerns related to formulation" and the "mathematical uncertainties" point to the TR's beliefs, whereas "decisions taken" indicate the self-regulation and the change the TR underwent in all the five subcategories when considering her AR development.

### *Developing skills for reaching a didactic or mathematical goal: Focusing on the problems posed*

**Table 2** presents an overall summary of the cognitive skills – divided into sub-categories – that the TR developed during the AR. Observing the progression teaches us about the TR's development as the four weeks progressed. Let us examine two aspects of the problem-posing process: the types of problems she learned to design and the skills she developed during the course of designing them.

#### *Types and pattern of problems.*

Problem-posing may be classified as "freely structured" (where the structure does not depend on constraints or style but depends on the situation), "semi-structured" (where the teacher is provided with an open-ended situation), or "goal-focused/structured" (where the problem is based on a specific didactic goal (Stoyanova & Ellerton, 1996; Abu-Elwan, 1999; Stoyanova, 2003).

In this study, the teacher progressed through all three stages. She first used a freely structured problem, taking inspiration from an existing idea. She later progressed to semi-structured, basing her questions on a specific part of the problem, such as the graph or part of formulation. Later, she realized that her goal was to elicit a very precise understanding in her students (why the derived graph has an asymptote), so she focused on this and designed a goal-focused problem aimed at a particular solution.

#### *Cognitive Skills developed*

Predicting perceptions and solutions that would be a result. Being able to predict which solutions students may come up with (both correct and incorrect) is an important skill for a teacher when posing problems. **Table 2** illustrates how the teacher's skills in prediction developed from week to week as she became more aware of various aspects of the problem as seen through her students' eyes: from modifying and adjusting a ready-made problem to, finally, formulating a problem specific ally to reach her mathematical or didactic goal.

In the first week, the TR posed the problem under the belief that she understood how students perceived graphs based on the attributes of functions. However, she failed to predict all the ways her students "understand" functions, as, contrary to her expectations, they responded with algebraic answers, not graphs. In problem 2, however, she considered her student's perceptions of roots and exponential functions and anticipated errors they might have in their visualization of the graphs or when they plotted them based on calculations. Therefore, she chose a problem to develop these perceptions and offer more insight into exponential functions. Even more so, she predicted that some students might try to give algebraic solutions, so she precisely worded the problem to circumvent this.

**Table 2.** TR's evolving skills and heuristic development

| Skill / Task | Modifying a given problem to achieve the same goal | Modifying a given problem to achieve different goal | Posing a new problem based on an existing graph\goal\theory | Posing a problem to understand students' perceptions of a feature or concept of graphs | Posing a problem to change students' perceptions about a feature or concept on graphs | Accuracy of problem formulation to achieve goals | Heuristic process |
|---|---|---|---|---|---|---|---|
| **Task 1** | Changing dynamic problem to static one. | Not found. | Not found. | The hidden goal was not explicitly formulated by her. | Not found. | Not found. | Solved a problem and identified that her students focused on the algorithmic aspect and her problem did not promote understanding the features of the function or graphing related to the algebraic features. Targeting was to attain a particular solution and What-If-Notting. |
| **Task 2** | | Schematic of an exponential function when the exponent has a root. | Not found. | Stated this goal. | This was a hidden goal not explicitly formulated by her. | Changed the wording several times, each time anticipating student solutions. | Solved a problem and discovered features she did not teach. So, she posed problem according to them. Targeting was to attain a particular solution and What-If-Notting. |
| **Task 3** | | | Found a graph and thought that students should explore its properties as a function and as a derivative. Learned theories about graphical thinking. | Stated this goal. | Stated a goal to change the misconception she identified. | Changed the wording several times, each time anticipating student solutions. | Did not think of the problem as a solver. Chaining & targeting of a particular solution |
| **Task 4** | | | Intended to develop a generalization of the concept of a relationship between a function graph and a derivative graph Learned theories about graphical thinking. | Stated this goal. | Stated a goal to change the misconception she identified. | Already adept at the proper formulation. | Knew how to distinguish between the characteristics of her solution and those of her students. What-If-Notting & targeting a particular solution. |

With problem 3, she started out with a precise mathematical goal (to draw a derivative graph using considerations about its domain) and explored how to achieve that specific goal. As a result, she found (in a textbook) an appropriate graph and assigned a new problem to it, predicting her students' solutions. With problem 4, she had finally developed to the stage where and could implement her new insight: that more than one problem must be solved for any particular topic to ensure that her students correctly understand that concept, and to ensure that her interpretation of the solutions (that is, her students' understanding) is based on more than one example. Therefore, she formulated a problem with several similar tasks for different functions, all of which led to understanding similar properties of a graph (i.e., the connection between a function's features and how their derivatives are graphed.

### *The heuristics of problem posing*

Kontorovich and Koichu (2009) and Schoenfeld (1985) have suggested that some problem-posing heuristics, such as generalization, problem decomposition or model creation, are also problem-solving heuristics. Heuristics that come from PP research include *systematic variation* (Silver et al., 1996), *What-If-Notting* (Brown & Walter, 1983), *chaining* (Silver et al., 1996), and *targeting* a particular solution (Koichu, 2008). Systematic variation is creating a new problem on the basis of a given or previously posed problem where one critical aspect of the problem is held constant. This was what happened in week 1 posing. The famous "What-If-Notting" can be seen as a systematic variation governed by a specific question. This is what happened in week 2.

Chaining refers to creating a new problem based on an answer (or an element of the solution), and this occurred in week 3 (the graph she found was the basic source). Targeting a particular solution refers to the PP process that is governed by one's decision to appropriate the problem formulation to a particular theorem, solution, or mathematical approach, which is what happened with problems three and four.

The heuristic used by the TR at the beginning of the project was that of a problem solver: thus, her choice of problems to pose in week 1 and 2. However, as her understanding and insight into her students' perceptions increased, her heuristic became that of a problem poser.

By focusing on targeting on how to ensure that her students arrive at the particular solution she was aiming for, she had to consider the "aptness to the potential solvers of a posed problem," as defined by Kontorovich and Koichu (2009). This – focusing on the suitability of the problem for her students − now became the exclusive heuristic in her PP.

From **Table 2** we can infer that the AR process was a good way to practice PP. The research encouraged the teacher to set a goal that was appropriate and essential for what she wanted to attain in her teaching. The traditional overall structure of an AR − planning, implementation, observation, and reflection (Gilbert & Newberry, 2004) − promotes the skills required. During the planning stage, the TR set sub-goals and posed problems accordingly. During the implementation stage, she built a lesson around the problem. She then carefully observed how her students solved the problems and analyzed what perceptions her students displayed, leading her to understand that she needed to elicit a broader concept of graphing skills in her students to allow them to solve the problem. Finally, during the reflection stage, she reviewed all the aspects, to decide whether the goal she had set was accurate enough and whether her wording of the problem led to that goal.

By following this circular process of an action research, every problem the teacher posed moved her closer to achieving the mathematical goals she had initially set. This process served to develop her professional perceptions of calculus graphing in the way stated by Lee, Capraro and Capraro (2018), and also her skills as a problem poser who can act to focus on targeting a particular solution.

## Conclusions

Laudonia et al. (2018) noted the characteristics of different modes of action research: technical action research, interactive action research, and teacher-centered action research. Using this division, the present study may be defined as interactive action research since the research was jointly negotiated by the teacher and an external expert (i.e. the mentor) and was, for the most part, carried out following interaction between the two. This interaction included examining the motives behind the AR project and noting that it included researching the students' cognition about creating graphs. In addition, AR projects aimed at gleaning insight into science students' cognition generally focus on knowledge development and attitude change among learners (Laudonia et al., 2018).

Laudonia and colleagues (2018) also noted that an AR project leads to professional development, even though − as in this particular case − professional development is not the goal itself. Mamlok-Naaman and colleagues (2013) also argued that teachers who perform AR undergo a process of professional change and growth. Briscoe and Wells (2002) theorized that AR promotes a teacher's professional development through a sequence of three basic steps: i) allowing the teacher to be willing to "risk change"; ii) encouraging the teacher to continuously reflect upon and assess her development; and iii) making her aware of the changes taking place − not only for her as a teacher but for her students as well.

All three steps were observed in this study: i) the TR was willing to "risk change" regarding researching the graphing concept in calculus that occurred as a result of the problems she solved and gained new insights; ii) she continuously reflected upon her problem-posing development for a specific purpose, and iii) from the second lesson on, she was aware of the changes in perception taking place both for her and for her students.

The teacher's professional development occurred in parallel to her students'. In fact, her potential for development depended on her understanding of her students' graphing perceptions. Stern (2014) claims there are three attributes required for proper AR: systematic operation conducted by teachers and researchers who form a coherent community that bases its performance on current theoretical methodology, self-reflection, and collaboration. In the present study, the author (mentor) and the teacher were connected to the community through collaborative learning about action research and about students' perceptions of the concepts of the graphs in the context of calculus. Both the author and teacher were involved in the reflective processes as they wrote in their journals and by conducting reflective discussion sessions throughout the process. Collaboration was expressed by the TR listening to the mentor's ideas and the mentor helping the TR draw accurate conclusions (for example, from examining the students' solutions).

As Davis, Clayton and Broome (2018) have stated, a feeling of success during an AR will influence the TR and have a positive impact on her teaching in practice. Here, teaching was actually a part of the study, and the actions that the teacher took during the lessons in which the students solved the lessons she designed were done with confidence that their planning was intensive and were the result of correct decisions she had made concerning various situations that the teacher and the mentor predicted. Therefore, the TR certainly felt that her teaching was organized and that she had successfully tracked the development of the concepts that she was trying to instill in her teaching.

This study completes some of the suggestions offered by previous researchers. For example, Cai et al. (2020), emphasized the necessity of organizing problem-posing workshops to develop teachers' problem-posing skills and to show them how to design lessons and courses based on the problem-posing approach. Also, Leavy and Hourigan (2019) found that the participation of prospective middle-school mathematics teachers in training related to PP and instruction had positive effects on the teachers' problem-posing skills.

This study adds to the aforementioned studies by emphasizing the five stages necessary during the problem-posing AR and the stages and heuristics in developing and posing problems for a particular mathematical purpose.

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